Ch. 12. Multiple Integration

12.1 Double Integration over Rectangular Regions

For a function f defined on a closed, bounded rectangular region R in the xy-plane, the double integral of f over R is given by

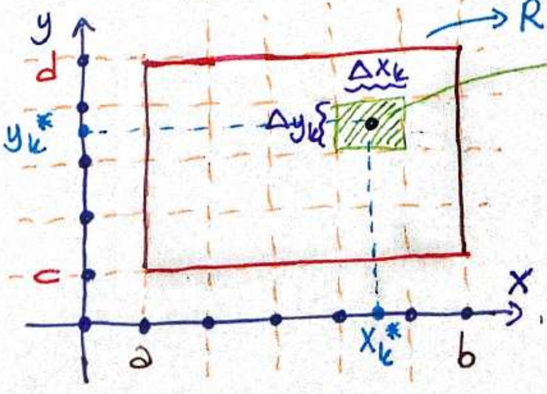
double integral of f over the rectangle R

Riemann sum

$$\boxed{\iint_R f(x,y)\,dA = \lim_{\|P\| \to 0} \sum_{k=1}^{N} f(x_k^*, y_k^*) \cdot \Delta A_k}$$

if this limit exists. In this case, f is integrable over R.

- R: A closed bounded rectangular region in the xy-plane defined by $a \le x \le b$ and $c \le y \le d$.
- A: The area of the rectangle R.
- P: A partition of the rectangle R into $N = m \cdot n$ subrectangles (cells), where m is the number of subintervals along the x-axis and n is the number along the y-axis.
- A_k: The area of the k-th subrectangle (cell) in the partition. ($A_k = \Delta x_k \cdot \Delta y_k$)
- x_k^*, y_k^*: Representative points chosen from each subrectangle in the partition.



A partition of the rectangle R into mn subrectangles, highlighting the representative of the k-th cell.

Properties of Double Integrals

1. $\forall a, b \in \mathbb{R}$, $f = f(x,y)$, $g = g(x,y)$,

$$\boxed{\iint_R (af + bg)\, dA = a \iint_R f\, dA + b \iint_R g\, dA.}$$ (linearity)

2. If $f(x,y) \geq g(x,y)$ in region R, then

$$\boxed{\iint_R f(x,y)\, dA \geq \iint_R g(x,y)\, dA.}$$ (dominance)

3. For subregions R_1 and R_2:

$$\boxed{\iint_R f(x,y)\, dA = \iint_{R_1} f(x,y)\, dA + \iint_{R_2} f(x,y)\, dA.}$$ (subdivision)

Region R

$R = R_1 \cup R_2$

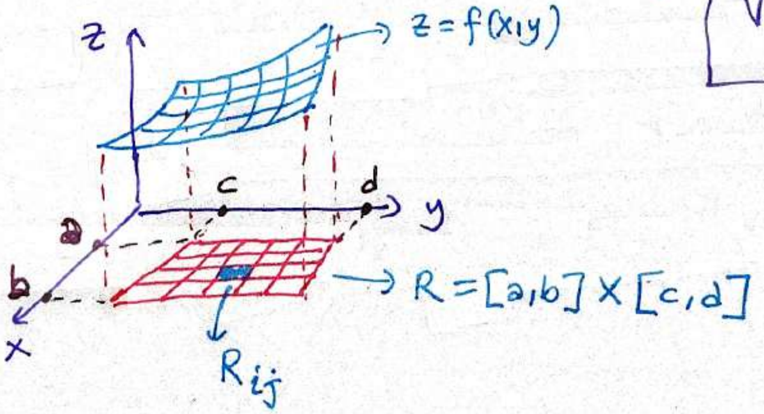
$$\boxed{V = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A}$$

volume from $z=0$ to $z=f(x,y)$

$$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A$$

$$\boxed{V = \iint_R f(x,y)\, dA}$$

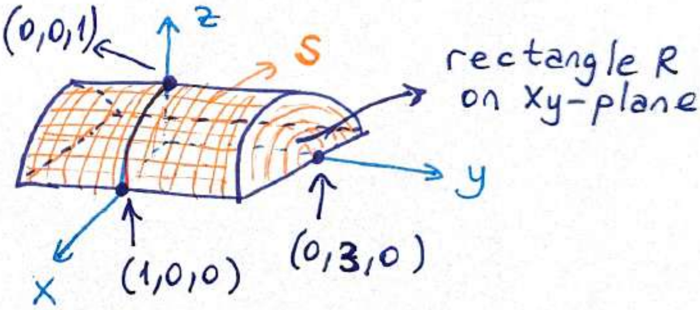
$f(x,y) \geq 0$



Ex. If $R = \{(x,y) \in \mathbb{R}^2 \mid -1 \leq x \leq 1, -3 \leq y \leq 3\}$, calculate

$$\iint_R \sqrt{1-x^2}\, dA .$$

Since $1-x^2 \geq 0$ we can interpret the integral as a volume. If $z = \sqrt{1-x^2}$ then $x^2+z^2=1$ and $z \geq 0$, so the double integral represents the volume of the solid S, under the circular cylinder $x^2+z^2=1$ and above the rectangle R.



The volume of the S is the product of the area of the semicircle with radius 1 $(r=1)$ and the height $(h=6)$ of the cylinder. Then,

$$\iint_R \sqrt{1-x^2}\, dA = \frac{1}{2} \cdot \pi \cdot (1)^2 \cdot (6) = 3\pi .$$

Iterated Integration

Multiple integrals are used to calculate the total value of functions with more than one variable over a spesific region. A double integral integrates a two-variable function $f(x,y)$ over a two-dimensional region.

$$\boxed{\iint_R f(x,y)\, dA = \underbrace{\int_a^b \underbrace{\int_c^d f(x,y)\, dy}_{\text{inner integral}}\, dx}_{\text{outer integral}} = \underbrace{\int_c^d \underbrace{\int_a^b f(x,y)\, dx}_{\text{inner integral}}\, dy}_{\text{outer integral}}}$$

(Fubini-Tonelli)

$(a,b,c,d \in \mathbb{R})$

If $R=[a,b]\times[c,d]$, $f(x,y)=\alpha(x).\beta(y)$, then

practical identity
$$\boxed{\int_a^b\int_c^d \alpha(x).\beta(y)\,\underset{\underset{dydx}{\parallel}}{dA} = \int_a^b \alpha(x)\,dx \,.\int_c^d \beta(y)\,dy\,.}$$

Ex.
$$\int_0^2\int_0^5 (3+y)\,dydx = \int_0^2 \underset{\underset{1}{\parallel}}{\alpha(x)}dx.\int_0^5 \underset{\underset{3+y}{\parallel}}{\beta(y)}dy$$
$$= \left(x\Big|_0^2\right)\cdot\left(\left(3y+\frac{y^2}{2}\right)\Big|_0^5\right)$$
$$= (2-0).\left[\left(3(5)+\frac{(5)^2}{2}\right)-(0)\right]$$
$$= 2.\left(15+\frac{25}{2}\right) = 2\,\frac{55}{2} = 55.$$

Ex. Evaluate $\iint_R x^3y^4\,dA$, where $R=\{(x,y)\in\mathbb{R}^2,\ 2\leq x\leq 3,\ 1\leq y\leq 2\}$

with
a) y-integration first, b) x-integration first.

a) $\displaystyle\int_2^3\left(\underbrace{\int_1^2 x^3y^4dy}\right)dx$ (keeping x constant)

$$\int_1^2 x^3y^4dy = x^3\int_1^2 y^4dy = x^3.\left(\frac{y^5}{5}\right)\Big|_1^2 = x^3\left(\frac{(2)^5}{5}-\frac{1}{5}\right)$$
$$= x^3.\left(\frac{32}{5}-\frac{1}{5}\right) = x^3.\,\frac{31}{5}$$

Then, integrate w.r.t. x:
$$\int_2^3 \frac{31}{5}x^3\,dx = \frac{31}{5}\int_2^3 x^3\,dx = \frac{31}{5}\left(\frac{x^4}{4}\right)\Big|_2^3 = \frac{31}{5}\left(\frac{(3)^4}{4}-\frac{(2)^4}{4}\right)$$
$$= \frac{31}{5}\left(\frac{81}{4}-\frac{16}{4}\right) = \frac{31}{5}\cdot\frac{65}{4} = \frac{403}{4}.$$

b) $\int_1^2 \left(\underbrace{\int_2^3 x^3 y^4 \, dx}_{} \right) dy \rightarrow$ (keeping y constant)

$$\int_2^3 x^3 y^4 \, dx = y^4 \int_2^3 x^3 \, dx = y^4 \left(\frac{x^4}{4} \right) \Big|_2^3 = y^4 \left(\frac{(3)^4}{4} - \frac{(2)^4}{4} \right)$$

$$= y^4 \left(\frac{81}{4} - \frac{16}{4} \right) = y^4 \cdot \frac{65}{4} .$$

Then, integrate w.r.t. y : (w.r.t. : with respect to)

$$\int_1^2 y^4 \, \frac{65}{4} \, dy = \frac{65}{4} \int_1^2 y^4 \, dy = \frac{65}{4} \left(\frac{y^5}{5} \right) \Big|_1^2 = \frac{65}{4} \left(\frac{(2)^5}{5} - \frac{(1)^5}{5} \right)$$

$$= \frac{65}{4} \left(\frac{32}{5} - \frac{1}{5} \right) = \frac{65}{4} \cdot \frac{31}{5} = \frac{403}{4} .$$

Ex. Find the following integrals

1) $\int_0^2 \int_0^4 x^2 y^6 \, dA$

2) $\int_{-1}^1 \int_{-2}^2 x y^3 \, dA$

3) $\int_0^1 \int_{-1}^3 x^3 y^4 \, dA$

4) $\int_{-1}^1 \int_{-3}^3 x y \, dA$

5) $\int_{-2}^1 \int_{-1}^2 x^2 y^2 \, dA$

6) $\int_0^1 \int_0^1 x^3 y^3 \, dA$

with

a) y integration first,

b) x integration first.

Ex. Find $\iint_R x \sin(xy)\, dA$ for

$$R = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le \pi,\ 0 \le y \le 1\}.$$

Let's first integrate w.r.t. x:

$$\int_0^1 \underbrace{\int_0^{\pi} x \sin(xy)\, dx\, dy}_{\text{inner integral}}$$

The inner integral requires integration by parts.

However, it is much simplier to first integrate w.r.t. y:

$$\int_0^{\pi}\int_0^1 x\sin(xy)\,dy\,dx = \int_0^{\pi} \left[-\frac{x\cos(xy)}{x}\right]\Big|_0^1 dx$$

$$= -\int_0^{\pi} \cos(xy)\Big|_0^1 dx$$

$$= -\int_0^{\pi} (\cos x - \cos 0)\, dx$$

$$= -\int_0^{\pi} (\cos x - 1)\, dx$$

$$= -(\sin x - x)\Big|_0^{\pi}$$

$$= (x - \sin x)\Big|_0^{\pi}$$

$$= (\pi - \sin\pi) - (0 - \sin 0)$$

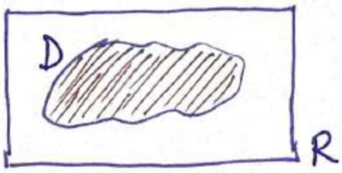
$$= (\pi - 0) - (0 - 0)$$

$$= \pi.$$

Ch.12.2 Double Integration over Nonrectangular Regions

Let $f(x,y)$ be continuous function on a region D inside a rectangle R. Define

$$F(x,y) = \begin{cases} f(x,y) & \text{for } (x,y) \text{ in } D \\ 0 & \text{for } (x,y) \text{ in } R\text{, but not in } D. \end{cases}$$



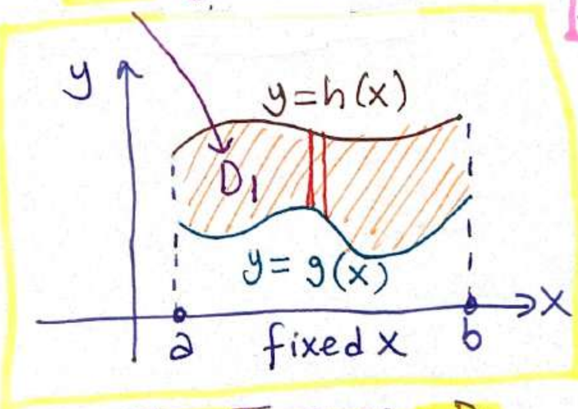
If F is integrable over R, then f is integrable over D.

$$\boxed{\iint_D f(x,y)\,dA = \iint_R F(x,y)\,dA.}$$

Even with boundary discontinuities, if f is continuous and the boundary well behaved, the double integral of f over D exists. This method applies to type I and type II regions.

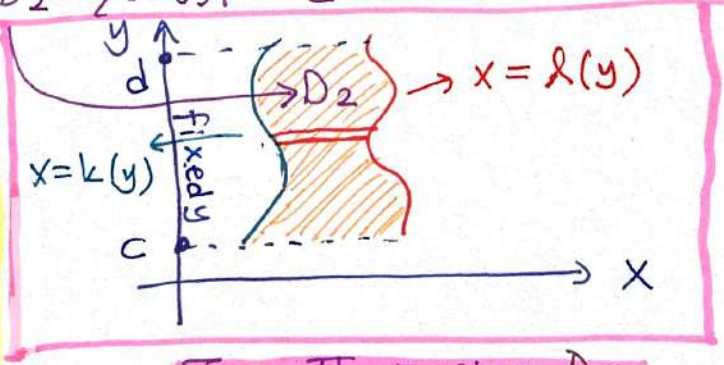
$\boxed{D_1 = \{(x,y) \mid x \in [a,b], y \in [g(x), h(x)]}$ and

$\boxed{D_2 = \{(x,y) \mid x \in [k(y), \ell(y)], y \in [c,d]\}}$



Type I region D_1

(consider vertical strip)



Type II region D_2

(consider horizontal strip)

For type I:

$$\iint_{D_1} f(x,y)\,dA = \iint_R F(x,y)\,dA$$

$$= \int_a^b \left[\int_c^d F(x,y)\,dy \right] dx$$

$$\iint_{D_1} f(x,y)\,dA = \int_a^b \int_{g(x)}^{h(x)} f(x,y)\,dy\,dx .$$

Similarly, for type II:

$$\iint_{D_2} f(x,y)\,dA = \iint_R F(x,y)\,dA$$

$$= \int_c^d \left[\int_a^b F(x,y)\,dx \right] dy$$

$$\iint_{D_2} f(x,y)\,dA = \int_c^d \int_{k(y)}^{\ell(y)} f(x,y)\,dx\,dy .$$

Then,

Fubini's theorem for nonrectangular regions:

$$\iint_{D_1} f(x,y)\,dA = \int_a^b \int_{g(x)}^{h(x)} f(x,y)\,dy\,dx ,$$ (Type I)

$$\iint_{D_2} f(x,y)\,dA = \int_c^d \int_{k(y)}^{\ell(y)} f(x,y)\,dx\,dy .$$ (Type II)

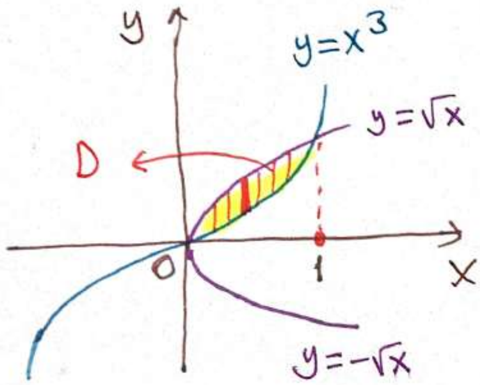
whenever integrals exist.

Ex. Evaluate the double integral

$$\int_0^1 \int_{x^3}^{\sqrt{x}} 120xy^2 \, dy\, dx .$$

Integrate w.r.t. y :

$$\int_{x^3}^{\sqrt{x}} 120xy^2\,dy = \int_{x^3}^{\sqrt{x}} 120x.\frac{y^3}{3}\Big|_{x^3}^{\sqrt{x}}$$



$$= 40x.(y^3)\Big|_{x^3}^{\sqrt{x}}$$

$$= 40x.\left[(\sqrt{x})^3 - (x^3)^3\right]$$

$$= 40x\left(x^{3/2} - x^9\right)$$

$$= 40x^{5/2} - 40x^{10}.$$

Then,

$$\int_0^1 \left(40x^{5/2} - 40x^{10}\right)dx = 40\int_0^1 x^{5/2}\,dx - 40\int_0^1 x^{10}\,dx$$

$$= 40\left(\frac{2}{7}x^{7/2}\right)\Big|_0^1 - 40\left(\frac{x^{11}}{11}\right)\Big|_0^1$$

$$= 40.\frac{2}{7} - 40.\frac{1}{11}$$

$$= 40\left(\frac{2}{7} - \frac{1}{11}\right) = 40\left(\frac{22-7}{77}\right) = 40.\frac{15}{77}$$

$$= \frac{600}{77}.$$

$$D = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; x \in [0,1],\ y \in [x^3, \sqrt{x}] \right\}$$

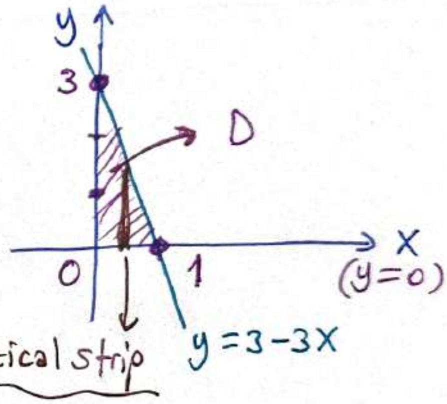
Ex. Evaluate the double integral

$$\iint_D (x^2+y)\,dA$$

over a triangular region $D = \{(x,y)\in\mathbb{R}^2 \mid x\in[0,1],\ y\in[0,3-3x]\}$ using iterated integrals with:

a) y-integration first,

b) x-integration first.



a) $$\int_0^1 \underbrace{\int_0^{3-3x} (x^2+y)\,dy}_{\text{inner integral}}\,dx,$$

$$\int_0^{3-3x} (x^2+y)\,dy = \int_0^{3-3x} \left(x^2y + \frac{y^2}{2}\right)\Big|_0^{3-3x}.$$

$$= \left[x^2(3-3x) + \frac{(3-3x)^2}{2}\right] - 0$$

$$= 3x^2 - 3x^3 + \frac{1}{2}\left(9 - 18x + 9x^2\right)$$

$$= 3x^2 - 3x^3 + \frac{9}{2} - 9x + \frac{9}{2}x^2$$

$$= \left(3+\frac{9}{2}\right)x^2 - 3x^3 - 9x + \frac{9}{2}$$

$$= -3x^3 + \frac{15}{2}x^2 - 9x + \frac{9}{2}.$$

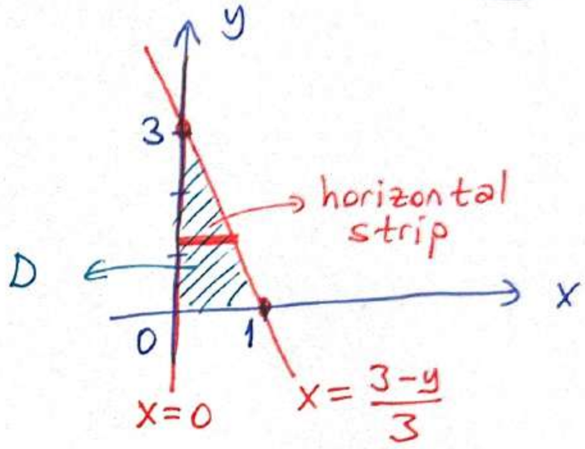
$$\Rightarrow \int_0^1 \left(-3x^3 + \frac{15}{2}x^2 - 9x + \frac{9}{2}\right)dx$$

$$= \left(-\frac{3}{4}x^4 + \frac{15}{2}\frac{x^3}{3} - 9\frac{x^2}{2} + \frac{9}{2}x\right)\Big|_0^1$$

$$= \left(-\frac{3}{4}(1)^4 + \frac{5}{2}(1)^3 - \frac{9}{2}(1)^2 + \frac{9}{2}(1)\right) - (0)$$

$$= -\frac{3}{4} + \frac{5}{2} \cancel{-\frac{9}{2}} \cancel{+\frac{9}{2}} = \frac{-3+10}{4} = \frac{7}{4}.$$

b) $$\underbrace{\int_0^3 \int_0^{\frac{3-y}{3}} (x^2+y)\,dx}_{\text{inner integral}}\,dy$$



$$\Rightarrow \int_0^{\frac{3-y}{3}} (x^2+y)\,dx$$

$$= \left(\frac{x^3}{3} + xy\right)\Big|_0^{\frac{3-y}{3}}$$

$$= \left[\frac{\left(\frac{3-y}{3}\right)^3}{3} + \left(\frac{3-y}{3}\right)\cdot y\right] - 0$$

$$= \frac{(3-y)^3}{81} + \frac{3y-y^2}{3}.$$

$$\Rightarrow \int_0^3 \left[\frac{(3-y)^3}{81} + y - \frac{1}{3}y^2\right] dy$$

$$= \left[-\frac{1}{81}\frac{(3-y)^4}{4} + \frac{y^2}{2} - \frac{y^3}{9}\right]\Big|_0^3$$

$$= \left[-\frac{1}{4\cdot 81}(3-(3))^4 + \frac{(3)^2}{2} - \frac{1}{9}(3)^3\right] - \left[-\frac{1}{81}\frac{(3-(0))^4}{4} + 0 - 0\right]$$

$$= \left[\quad 0 \quad + \frac{9}{2} - \frac{27}{9}\right] + \frac{1}{\cancel{81}}\cdot\frac{\cancel{81}}{4}$$

$$= \frac{9}{2} - 3 + \frac{1}{4} = \frac{18-12+1}{4} = \frac{7}{4}.$$

Double Integral as Area and Volume

The area of region D can be computed using

$$\boxed{A = \iint_D dA}$$

{ A : area }

This method is often easier than using single integrals.

For a non-negative continuous function $f(x,y)$ over region D, the volume of the solid of the surface $z = f(x,y)$ is determined by

$$\boxed{V = \iint_D f(x,y)\, dA}$$

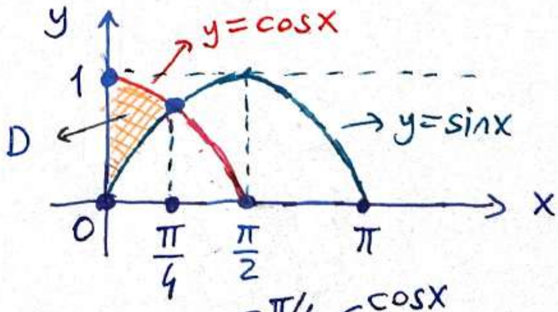
{ V: volume }

Ex. (from the textbook):

Find the area of the region D between $y = \sin x$ and $y = \cos x$ over $x \in [0, \frac{\pi}{4}]$ using

a) a single integral,

b) a double integral.



a) $A = \int_0^{\pi/4} (\cos x - \sin x)\, dx$

$= (\sin x + \cos x)\Big|_0^{\pi/4}$

$= \left(\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\right) - (\sin 0 + \cos 0)$

$= \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - (0 + 1)$

$= \sqrt{2} - 1$.

b) $A = \iint_D dA = \int_0^{\pi/4} \int_{\sin x}^{\cos x} 1 . dy\, dx$

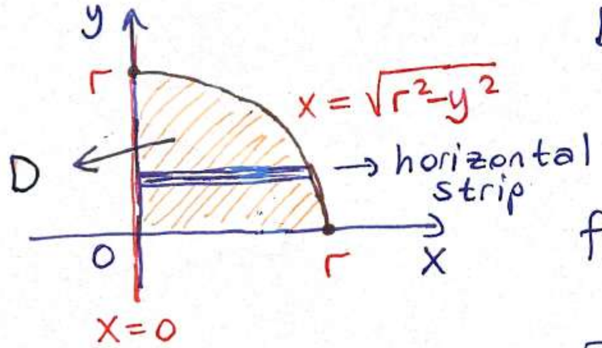
$= \int_0^{\pi/4} y \Big|_{\sin x}^{\cos x} dx$

$= \int_0^{\pi/4} (\cos x - \sin x)\, dx$

$= (\sin x + \cos x)\Big|_0^{\pi/4}$

$= \sqrt{2} - 1$.

Ex. Determine the volume of the solid bounded above by the plane $y=z$ and below in the xy-plane by the region defined as the portion of the disk $x^2+y^2 \le r^2$ located in the first quadrant. $(r>0)$



$$D = \{ (x,y) \in \mathbb{R}^2 \mid x \in [0, \sqrt{r^2-y^2}],\ y \in [0, r] \}$$

$$f(x,y) = y \ , \quad dA = dx\,dy$$

$$V = \iint_D f(x,y)\, dA = \int_0^r \int_0^{\sqrt{r^2-y^2}} y\, dx\, dy$$

$$= \int_0^r y x \Big|_0^{\sqrt{r^2-y^2}} dy$$

$$= \int_0^r y\left(\sqrt{r^2-y^2} - 0\right) dy$$

$$= \int_0^r y\sqrt{r^2-y^2}\, dy \ , \quad \left\{ \begin{array}{l} u = r^2-y^2 \\ du = -2y\,dy \end{array} \right\}$$

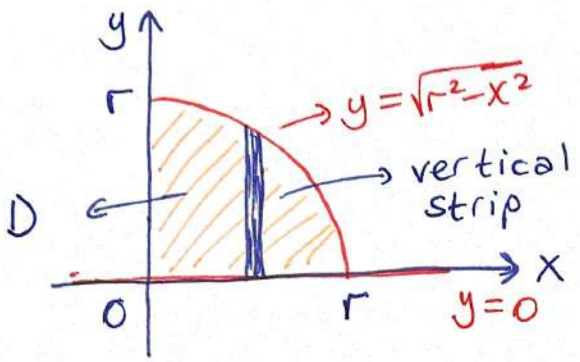
$$= \left[-\frac{1}{3}(r^2-y^2)^{3/2} \right] \Big|_0^r$$

$$= -\frac{1}{3}\left[(r^2-r^2)^{3/2} - (r^2-0^2)^{3/2} \right]$$

$$= -\frac{1}{3}\left(0 - r^3\right)$$

$$V = \frac{r^3}{3}$$

or taking $D=\{(x,y)\in\mathbb{R}^2 \mid x\in[0,r],\ y\in[0,\sqrt{r^2-x^2}]\}$



$f(x,y)=y$,

$dA=dy\,dx$,

$$V=\iint_D f(x,y)\,dA=\int_0^r\int_0^{\sqrt{r^2-x^2}} y\,dy\,dx$$

$$=\int_0^r \frac{y^2}{2}\Big|_0^{\sqrt{r^2-x^2}}dx$$

$$=\int_0^r\left[\frac{(\sqrt{r^2-x^2})^2}{2}-\frac{(0)^2}{2}\right]dx$$

$$=\int_0^r \frac{r^2-x^2}{2}\,dx$$

$$=\frac{1}{2}\left(r^2x-\frac{x^3}{3}\right)\Big|_0^r$$

$$=\frac{1}{2}\left[\left(r^3-\frac{r^3}{3}\right)-\left(r^2(0)-\frac{(0)^3}{3}\right)\right]$$

$$=\frac{1}{2}\left(\frac{2r^3}{3}\right)$$

$$V=\frac{r^3}{3}.$$

When evaluating the integral $\iint_D f(x,y)\,dA$, you can choose to integrate w.r.t. x or y first. In the following you will change the given order of integration.

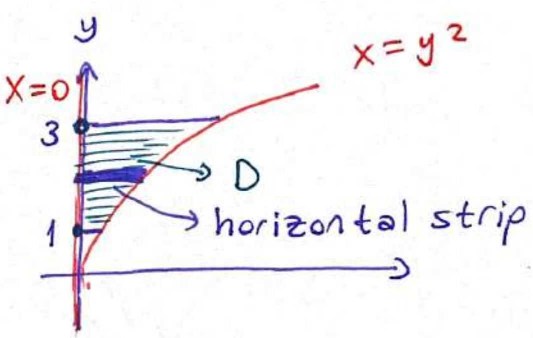
Ex. Change the order of integration in the iterated

$$\int_1^3 \int_0^{y^2} f(x,y)\,dx\,dy$$

y varies from 1 to 3

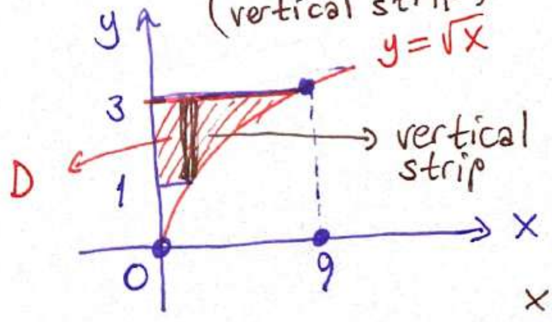
x varies from 0 to y^2

Type II: y fixed (consider horizontal strip)



$x = y^2 \Rightarrow y = \sqrt{x}$. (For Type I)

For a fixed x, y will range from $\sqrt{x}$ to 3.
(consider vertical strip)



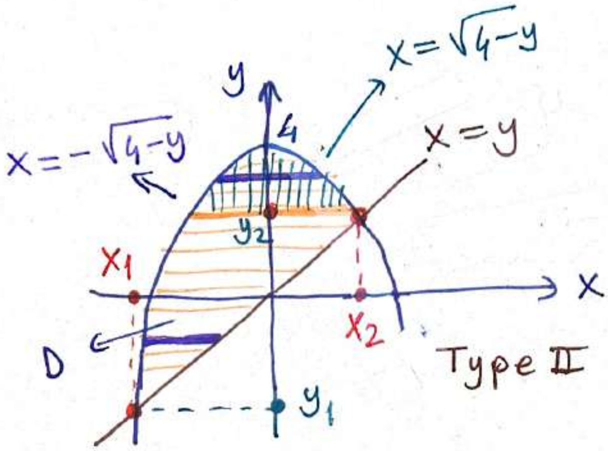
The reversed order of integration will be

$$\int_0^9 \int_{\sqrt{x}}^3 f(x,y)\,dy\,dx.$$

x varies from 0 to 9

y varies from $\sqrt{x}$ to 3

Ex. Consider the region D bounded by parabola $y = 4 - x^2$ and the line $y = x$. Determine whether to use a Type I or Type II description to find the area of D.

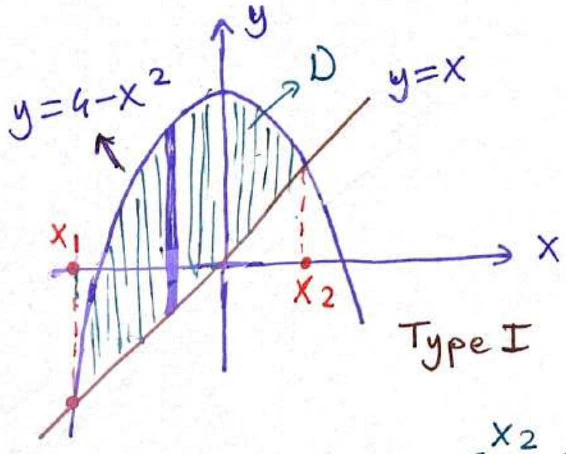


$$4-x^2=x$$

$$x^2+x-4=0$$

$$x_{1,2}=\frac{-1\mp\sqrt{1^2-4.1.(-4)}}{2.1}=\frac{-1\mp\sqrt{17}}{2}$$

$x_1 \approx -2.56$, $x_2 \approx 1.56$



Type I: $A = \int_{x_1}^{x_2}\int_{x}^{4-x^2} dy\,dx$

Type II: $A = \int_{y_1}^{y_2}\int_{y}^{-\sqrt{4-y}} dx\,dy + \int_{y_2}^{4}\int_{-\sqrt{4-y}}^{\sqrt{4-y}} dx\,dy$

Here, the convenient of computation is clearly demonstrated in Type I.

$$A = \int_{x_1}^{x_2}\int_{x}^{4-x^2} dy\,dx = \int_{x_1}^{x_2} y\Big|_{x}^{4-x^2} dx$$

$$= \int_{x_1}^{x_2}\left[4-x^2-x\right]dx = \left(4x-\frac{x^3}{3}-\frac{x^2}{2}\right)\Big|_{x_1}^{x_2}$$

$A \approx 11.68.$

Ex. Compute $\int_0^1 \int_x^1 e^{y^2} dy\,dx$ (from textbook)

The y integration cannot be done first since e^{y^2} lacks an elementary antiderivative. Instead, we will reverse the order of integration. For any fixed x between 0 and 1, y ranges from x to 1.

To change the order of integration, observe that for each fixed y between 0 and 1, x varies from 0 to y.

$$\int_0^1 \int_x^1 e^{y^2} dy\,dx = \int_0^1 \int_0^y e^{y^2} dx\,dy$$

$$= \int_0^1 x \cdot e^{y^2} \Big|_0^y \, dy$$

$$= \int_0^1 \left[(y)e^{y^2} - (0)e^{y^2}\right] dy$$

$$= \int_0^1 y e^{y^2} dy \quad \left(u = y^2,\ du = 2y\,dy\right)$$

$$= \frac{1}{2} e^{y^2} \Big|_0^1$$

$$= \frac{1}{2}\left(e^1 - e^0\right)$$

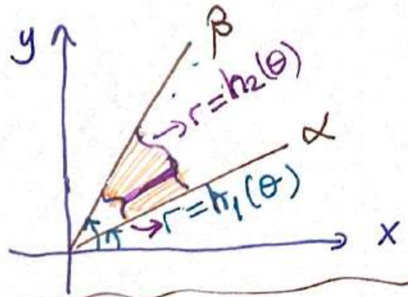
$$= \frac{1}{2}(e-1).$$

12.3 Double Integrals in Polar Coordinates

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ r &= \sqrt{x^2+y^2} \\ \tan\theta &= \frac{y}{x} \end{aligned}\right\}$$

If $f(x,y) = x^2+y^2+a^2$, then

$$f(r\cos\theta, r\sin\theta) = r^2+a^2.$$



$$\boxed{V = \iint_R f(x,y)\,dA = \iint_D f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta}$$

$$dx\,dy = r\,dr\,d\theta. \qquad \boxed{A = \iint_D r\,dr\,d\theta}$$

$$\boxed{\iint_D f(r,\theta)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r,\theta)\,r\,dr\,d\theta}$$

To transform the integral $\iint f(x,y)\,dA$ under a change of variables $x = x(u,v)$, $y = y(u,v)$, we get

$$\iint f(x,y)\,dA = \iint f(u,v)\,|J(u,v)|\,du\,dv,$$

where $\boxed{J(u,v) = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}}$. (Jacobian determinant)

For polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$,

$$J(r,\theta) = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta$$

$$= r.$$

$$\Rightarrow \boxed{\iint_R f(x,y)\,dA = \iint_D f(r,\theta)\,r\,dr\,d\theta = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta}$$

Ex. Evaluate $\iint_R (3x+4y^2)\,dA$, where R represents the region of the upper-half bounded by the circles $x^2+y^2=1$ and $x^2+y^2=4$.

$$R=\{(x,y)\in\mathbb{R}^2 \mid y\geq 0,\ 1\leq x^2+y^2\leq 4\}$$

$1\leq r\leq 2,\quad 0\leq\theta\leq\pi.$

$$\Rightarrow \iint_R (3x+4y^2)\,dA = \int_0^{\pi}\int_1^2 (3r\cos\theta+4r^2\sin^2\theta)\,r\,dr\,d\theta$$

$$=\int_0^{\pi}\int_1^2 (3r^2\cos\theta+4r^3\sin^2\theta)\,dr\,d\theta$$

$$=\int_0^{\pi}(r^3\cos\theta+r^4\sin^2\theta)\Big|_1^2\,d\theta$$

$$=\int_0^{\pi}\left[((2)^3\cos\theta+(2)^4\sin^2\theta)-((1)^3\cos\theta+(1)^4\sin^2\theta)\right]d\theta$$

$$=\int_0^{\pi}(7\cos\theta+15\sin^2\theta)\,d\theta$$

$$=\int_0^{\pi}\left[7\cos\theta+\frac{15}{2}(1-\cos 2\theta)\right]d\theta$$

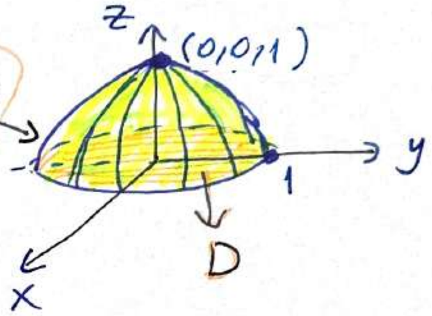
$$=\left(7\sin\theta+\frac{15}{2}\theta-\frac{15}{4}\sin 2\theta\right)\Big|_0^{\pi}$$

$$=\frac{15\pi}{2}$$

Ex. Find the volume of the solid bounded by the plane $z=0$ and the paraboloid $z=1-x^2-y^2$

$z=0 \Rightarrow x^2+y^2=1$ (circle)

region D $x^2+y^2\leq 1$



$0\leq r\leq 1$, $0\leq \theta \leq 2\pi$

(polar region) $D=\{(r,\theta)\mid 0\leq r\leq 1,\ 0\leq\theta\leq 2\pi\}$

$1-x^2-y^2 = 1-(x^2+y^2) = 1-r^2$

$$\Rightarrow V = \iint_D (1-x^2-y^2)\,dA = \int_0^{2\pi}\int_0^1 (1-r^2)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^1 (r-r^3)\,d\theta$$

$$= 2\pi\left(\frac{r^2}{2}-\frac{r^4}{4}\right)\Big|_0^1 = 2\pi\left(\frac{1}{2}-\frac{1}{4}\right) = 2\pi\,\frac{1}{4}$$

$$\Rightarrow \boxed{V=\frac{\pi}{2}}.$$

$D=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\leq 1\}$ (Cartesian region)

If we had used Cartesian coordinates, instead polar,

$$V=\iint_D (1-x^2-y^2)\,dA = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1-x^2-y^2)\,dy\,dx.$$

It would not be easy to evaluate, since it involved finding the integrals

$$\int\sqrt{1-x^2}\,dx\ ,\ \int x^2\sqrt{1-x^2}\,dx\ ,\ \int(1-x^2)^{3/2}\,dx.$$

Ex. Compute the area of the region R bounded by the circle $r=4$ and the line $\theta=\frac{\pi}{4}$.

For the circle: r ranges from 0 to 4,
the line : θ ranges from 0 to $\frac{\pi}{4}$.

$$A = \int_{\theta=0}^{\frac{\pi}{4}} \int_{r=0}^{4} r\,dr\,d\theta$$

$$= \int_{0}^{\pi/4} \frac{r^2}{2}\Big|_{0}^{4} d\theta = \int_{0}^{\pi/4} 8\,d\theta$$

$$= 8\int_{0}^{\pi/4} d\theta = 8(\theta)\Big|_{0}^{\pi/4} = 8.\frac{\pi}{4} = 2\pi.$$

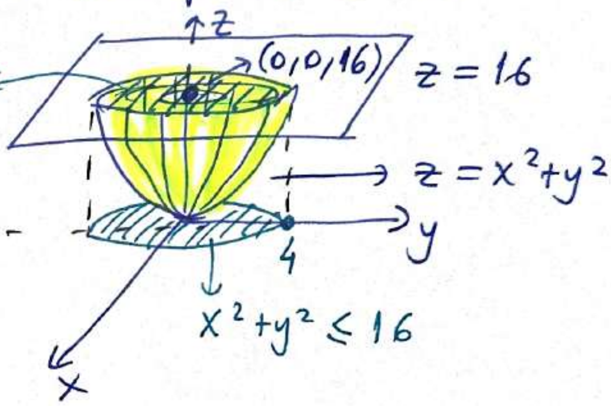
Ex. Compute the area of the region R bounded by the circle $x^2+y^2=16$ and the line $y=x$. (Use double integral and solve it by polar form.)

Ex. Evaluate $\int_{\theta=0}^{2\pi} \int_{r=0}^{3} r\,dr\,d\theta$.

Ex. Evaluate $\int_{0}^{1} \int_{0}^{\sqrt{2-x^2}} y\,dy\,dx$ by replacing to polar coordinates.

Ex. Evaluate $\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{0} \cos(x^2+y^2)\,dy\,dx$ by converting to polar coordinates.

Ex. Find the volume of the region that lies inside $z = x^2 + y^2$ and below the plane $z = 16$.
(Use polar form)



$$V = \iint_D 16 \, dA - \iint_D (x^2+y^2) \, dA$$

$$= \iint_D (16 - x^2 - y^2) \, dA$$

$\Rightarrow \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq r \leq 4, \quad z = 16 - r^2$

$$\Rightarrow V = \iint_D \left[16 - (x^2+y^2)\right] dA$$

$$V = \int_0^{2\pi} \int_0^4 r(16 - r^2) \, dr \, d\theta$$

$$V = \int_0^{2\pi} \left(8r^2 - \frac{1}{4} r^4\right) \Big|_0^4 \, d\theta$$

$$V = \int_0^{2\pi} 64 \, d\theta$$

$\Rightarrow V = 128\pi$

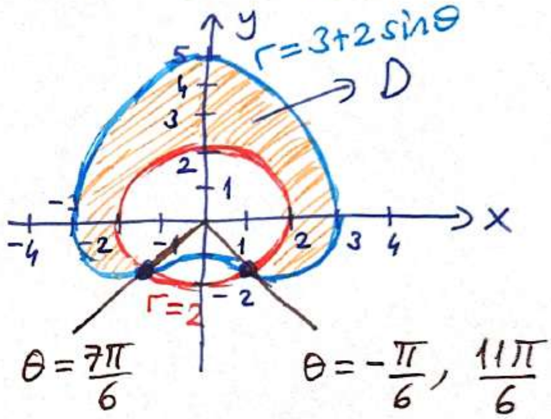
Ex. Compute the area of the region that lies inside $r=3+2\sin\theta$ and outside $r=2$.

$$3+2\sin\theta = 2$$

$$2\sin\theta = 2-3$$

$$\sin\theta = -\frac{1}{2}$$

$$\theta = \frac{7\pi}{6},\ \frac{11\pi}{6}$$



$$2 \le r \le 3+2\sin\theta$$

$$A = \iint_D dA = \int_{-\pi/6}^{7\pi/6}\int_2^{3+2\sin\theta} r\,dr\,d\theta$$

$$= \int_{-\pi/6}^{7\pi/6} \frac{r^2}{2}\Big|_2^{3+2\sin\theta}$$

$$= \int_{-\pi/6}^{7\pi/6}\left(\frac{5}{2}+6\sin\theta+2\sin^2\theta\right)d\theta$$

$$= \int_{-\pi/6}^{7\pi/6}\left[\frac{7}{2}+6\sin\theta-\cos(2\theta)\right]d\theta$$

$$= \left(\frac{7}{2}\theta - 6\cos\theta - \frac{1}{2}\sin(2\theta)\right)\Big|_{-\pi/6}^{7\pi/6}$$

$$= \frac{11\sqrt{3}}{2} + \frac{14}{3}\pi$$

$$\cong 24.19$$

Ex. Using the polar coordinates, compute the improper integral

$$\iint_{\mathbb{R}^2} e^{-3(x^2+y^2)}\,dx\,dy,$$

and then compute the following improper integral

$$\int_{-\infty}^{\infty} e^{-3x^2}\,dx.$$

In polar coordinates, $x^2+y^2=r^2$ and $dx\,dy = r\,dr\,d\theta$.

$$\Rightarrow \iint_{\mathbb{R}^2} e^{-3(x^2+y^2)}\,dx\,dy = \int_0^{2\pi}\int_0^{\infty} e^{-3r^2}\cdot r\,dr\,d\theta \quad \left\{\begin{array}{l}\text{convert to}\\ \text{polar coordinates}\end{array}\right\}$$

$$= \int_0^{2\pi} d\theta \cdot \int_0^{\infty} r\cdot e^{-3r^2}\,dr \quad \left\{\begin{array}{l}\text{seperate the}\\ \text{integrals}\end{array}\right\}$$

$$= 2\pi \int_0^{\infty} r\cdot e^{-3r^2}\,dr$$

$$= 2\pi \int_0^{\infty} e^{-u}\,\frac{du}{6} \quad \left\{\begin{array}{l} u = 3r^2 \\ du = 6r\,dr \end{array}\right\}$$

$$= \frac{\pi}{3}\int_0^{\infty} e^{-u}\,du \quad \{\text{improper integral}\}$$

$$= \frac{\pi}{3}\lim_{b\to\infty}\int_0^{b} e^{-u}\,du$$

$$\left\{ \int_0^b e^{-u}\,du = \left(-e^{-u}\right)\Big|_0^b = \left(-e^{-b}\right)-\left(-e^{0}\right) = -e^{-b}+1 = 1-\frac{1}{e^b} \right\}$$

$$\Rightarrow \boxed{\iint_{\mathbb{R}^2} e^{-3(x^2+y^2)}\,dx\,dy} = \frac{\pi}{3}\lim_{b\to\infty}\left(1-\frac{1}{e^b}\right) = \frac{\pi}{3}(1-0) = \frac{\pi}{3}$$

$$\Rightarrow \iint_{\mathbb{R}^2} e^{-3(x^2+y^2)}\,dx\,dy = \int_{-\infty}^{\infty} e^{-3x^2}\,dx \cdot \int_{-\infty}^{\infty} e^{-3y^2}\,dy = \left(\int_{-\infty}^{\infty} e^{-3x^2}\,dx\right)^2 = \frac{\pi}{3}$$

$$\Rightarrow \boxed{\int_{-\infty}^{\infty} e^{-3x^2}\,dx = \sqrt{\frac{\pi}{3}}}.$$

12.4 Surface Area

If f has a continuous derivative on the closed, bounded interval $[a,b]$, the length ℓ of the graph of $y=f(x)$, $x\in[a,b]$, is given by

$$\boxed{\ell = \int_a^b \sqrt{1+[f'(x)]^2}\, dx}.$$

Analogous to the length formula for a single-variable function we aim to explore a formula for the surface area of a differentiable function of two variables.

Surface Area as a Double Integral

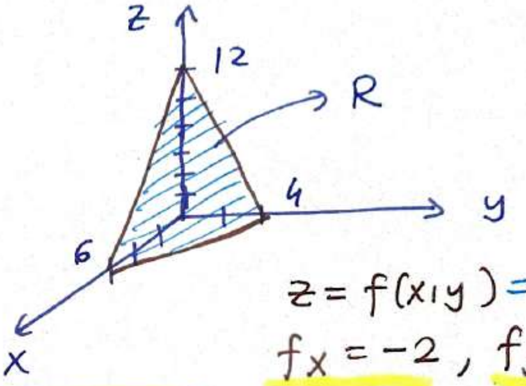
For a function $f(x,y)$ with continuous partial derivatives f_x and f_y in region R, the surface area S over R is given by

$$\boxed{S = \iint_R \sqrt{[f_x(x,y)]^2+[f_y(x,y)]^2+1}\; dA}$$

where $dS = \sqrt{(f_x)^2+(f_y)^2+1}\; dA$ is the surface area element.

$\int_a^b dx$: length on axis x.
$\int_a^b ds = \int_a^b \sqrt{[f'(x)]^2+1}\, dx$: arc length.
$\iint_R dA$: area in xy-plane.
$\iint_R dS = \iint_R \sqrt{[f_x(x,y)]^2+[f_y(x,y)]^2+1}\; dA$: surface area

Ex. Find the surface area of the part of the plane $2x+3y+z=12$ that lies in the first octant.



$z = 12-2x-3y$

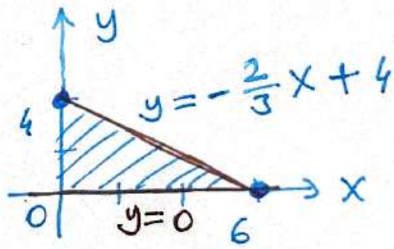
$(x,y)=(0,0) \Rightarrow z=12$

$(x,z)=(0,0) \Rightarrow y=4$

$(y,z)=(0,0) \Rightarrow x=6$

$z = f(x,y) = 12-2x-3y$

$f_x=-2, \quad f_y=-3$



$$\boxed{S = \iint_R \sqrt{(f_x)^2+(f_y)^2+1}\; dA}$$

$$= \int_0^6 \int_0^{-\frac{2}{3}x+4} \sqrt{(-2)^2+(-3)^2+1}\; dy\,dx$$

$$= \int_0^6 \int_0^{-\frac{2}{3}x+4} \sqrt{14}\; dy\,dx$$

$$= \int_0^6 \sqrt{14}\cdot(y)\Big|_0^{-\frac{2}{3}x+4} = \int_0^6 \sqrt{14}\cdot\left(-\frac{2}{3}x+4\right) dx$$

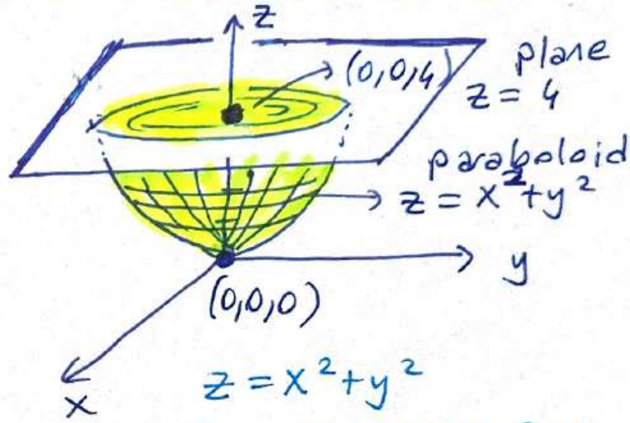
$$= \sqrt{14}\cdot\left(-\frac{2}{3}\frac{x^2}{2}+4x\right)\Big|_0^6$$

$$S = \sqrt{14}\left(-\frac{(6)^2}{3}+4\cdot(6)\right) = \sqrt{14}\left(-12+24\right) = 12\sqrt{14}.$$

Ex. Find the surface area of the part of the plane $x+y+z=1$ that lies in the first octant.

Ex. Find the surface area of the part of $z=xy$ that lies in the cylinder given by $x^2+y^2=1$.

Ex. Find the surface area of the portion of the paraboloid $x^2+y^2-z=0$ that lies below the plane $z=4$.



$z=x^2+y^2$

$z_x = 2x\,,\ z_y = 2y$

$$S = \iint_D \sqrt{1+(f_x)^2+(f_y)^2}\, dA$$

$$S = \iint_D \sqrt{1+(2x)^2+(2y)^2}\, dA$$

$$S = \iint_D \sqrt{1+4(x^2+y^2)}\, dA$$

$$S = \int_0^{2\pi}\int_0^{2} \sqrt{1+4r^2}\, r\,dr\,d\theta$$

$\left\{ \begin{array}{l} x=r\cos\theta \\ y=r\sin\theta \end{array} \right\}$ change to polar coordinates

$$= \int_0^{2\pi}\int_1^{17} \sqrt{u}\cdot\frac{du}{8}\, d\theta$$

$\left\{ \begin{array}{l} u=1+4r^2 \\ du = 8r\,dr \end{array} \right\}$ $\left\{ \begin{array}{l} r=0 \Rightarrow u=1 \\ r=2 \Rightarrow u=17 \end{array} \right\}$

$$= \int_0^{2\pi} \frac{1}{8}\frac{2}{3}\cdot\left(u^{3/2}\right)\Big|_1^{17}\, d\theta$$

$$= \int_0^{2\pi} \frac{1}{12}\left(17^{3/2}-1\right) d\theta$$

$$= \frac{1}{12}\left(17^{3/2}-1\right)(\theta)\Big|_0^{2\pi}$$

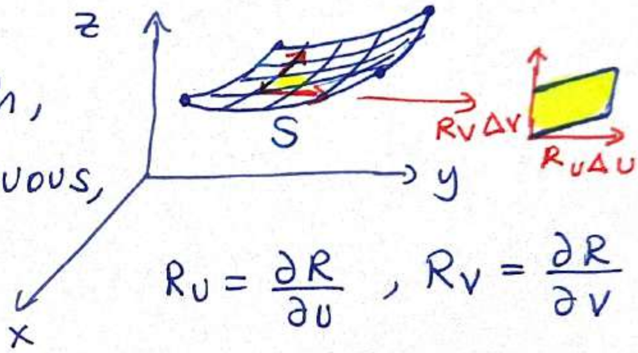
$$= \frac{1}{12}\left(17^{3/2}-1\right)\cdot(2\pi-0)$$

$$= \frac{\pi}{6}\cdot\left(17^{3/2}-1\right).$$

Area of Parametric Surface

Let S be a surface defined parametrically by
$R(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$ on the region D in the uv-plane.

Assume that S is smooth, meaning R_u and R_v are continuous, and $R_u \times R_v \neq 0$ on D.



$$R_u = \frac{\partial R}{\partial u}, \quad R_v = \frac{\partial R}{\partial v}$$

The surface area $\mathcal{S}$ is defined by

$$\boxed{\mathcal{S} = \iint_D \| R_u \times R_v \| \, du\, dv}, \quad \left\{ \begin{array}{l} R_u = \langle x_u, y_u, z_u \rangle \\ R_v = \langle x_v, y_v, z_v \rangle \end{array} \right\}$$

where $R_u \times R_v$ is named the fundamental cross product.

Ex. Find the surface area of the sphere with radius r using double integrals.

Parametric representation of the sphere with radius r:

$$\left. \begin{array}{l} x(u,v) = r \sin u \cos v \\ y(u,v) = r \sin u \sin v \\ z(u,v) = r \cos u \end{array} \right\} \quad \begin{array}{l} u \in [0, \pi], \\ v \in [0, 2\pi] \end{array}$$

$$D = \{ (u,v) \mid u \in [0,\pi], v \in [0, 2\pi] \}$$

$$R(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$$

$$\Rightarrow R(u,v) = \langle r\sin u \cos v, r \sin u \sin v, r \cos u \rangle$$

$$R_u = \frac{\partial R}{\partial u} = \langle r \cos u \cos v, r \cos u \sin v, -r \sin u \rangle$$

$$R_v = \frac{\partial R}{\partial v} = \langle -r \sin u \sin v, r \sin u \cos v, 0 \rangle$$

$$R_u \times R_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ r\cos u\cos v & r\cos u\sin v & -r\sin u \\ -r\sin u\sin v & r\sin u\cos v & 0 \end{vmatrix}$$

$$= \left[(r\cos u\sin v).(0) - (-r\sin u)(r\sin u\cos v)\right]\vec{i}$$
$$- \left[(r\cos u\cos v).(0) - (-r\sin u\sin v)(-r\sin u\right]\vec{j}$$
$$+ \left[(r\cos u\cos v)(r\sin u\cos v) - (-r\sin u\sin v)(r\cos u\sin v)\right]\vec{k}$$
$$= (r^2\sin^2 u\cos v)\vec{i} - (r^2\sin^2 u\sin v)\vec{j}$$
$$+ r^2(\cos u\sin u\cos^2 v + \sin u\cos u\sin^2 v)\vec{k}$$
$$= (r^2\sin^2 u\cos v)\vec{i} - (r^2\sin^2 u\sin v)\vec{j} + (r^2\cos u\sin u)\vec{k}$$

$$\|R_u \times R_v\| = \sqrt{r^4\sin^4 u\cos^2 v + r^4\sin^4 u\sin^2 v + r^4\cos^2 u\sin^2 u}$$
$$= \sqrt{r^4\sin^4 u(\cos^2 v + \sin^2 v) + r^4\cos^2 u\sin^2 u}$$
$$= \sqrt{r^4\sin^4 u + r^4\cos^2 u\sin^2 u}$$
$$= \sqrt{r^4\sin^2 u(\sin^2 u + \cos^2 u)} = r^2\sin u$$

$$S = \iint_D \|R_u \times R_v\|\, dA = \int_0^{2\pi}\int_0^{\pi} r^2\sin u\, du\, dv$$

$$S = r^2\int_0^{2\pi} dv \cdot \int_0^{\pi}\sin u\, du = r^2(2\pi).2 = 4\pi r^2.$$

Ex. Find the surface area of the surface

$$R(u,v) = (u\sin v)\vec{i} + (u\cos v)\vec{j} + u\vec{k}$$

for $u \in [0,4]$, $v \in [0, 2\pi]$.

12.5 Triple Integrals

For a function f defined over a closed bounded solid region D, the triple integral is given by

$$\boxed{\iiint_D f(x,y,z)\,dV = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\, V_k},$$

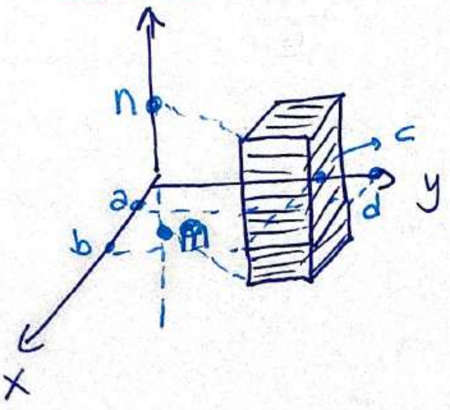
where the limit exists and V_k are the volumes of subregions in D.

In triple integrals, as in double integrals, the properties of linearity, dominance, and subdivision also apply.

Iterated Integration

Fubini's theorem over a parallelpiped in 3D states that if $f(x,y,z)$ is continuous over a rectangular box $B = \{(x,y,z) \in \mathbb{R}^3 \mid x \in [a,b], y \in [c,d], z \in [m,n]\}$, then the triple integral can be expressed as an iterated integral:

$$\boxed{\iiint_B f(x,y,z)\,dV = \int_m^n \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz}.$$



The integration can be carried out in any order, with adjustments to the differential elements as required, including $dx\,dy\,dz$, $dx\,dz\,dy$, $dz\,dx\,dy$, $dy\,dx\,dz$, $dy\,dz\,dx$ or $dz\,dy\,dx$.

Ex. Evaluate $\iiint_B xy^2z\,dV$, where

$B = \{(x,y,z) \in \mathbb{R}^3 \mid x \in [0,1],\ y \in [-1,1],\ z \in [2,3]\}$.

$$\iiint_B xy^2z\,dz\,dy\,dx = \int_0^1 \int_{-1}^1 \int_2^3 xy^2z\,dz\,dy\,dx$$

$$= \int_0^1 \int_{-1}^1 xy^2\left(\frac{z^2}{2}\right)\Big|_2^3\,dy\,dx$$

$$= \int_0^1 \int_{-1}^1 \frac{xy^2}{2}\left((3)^2 - (2)^2\right)dy\,dx$$

$$= \frac{5}{2}\int_0^1 x \cdot \frac{(y^3)}{3}\Big|_{-1}^1\,dx$$

$$= \frac{5}{6}\int_0^1 x\left[(1)^3 - (-1)^3\right]dx$$

$$= \frac{5}{6}\int_0^1 2x\,dx$$

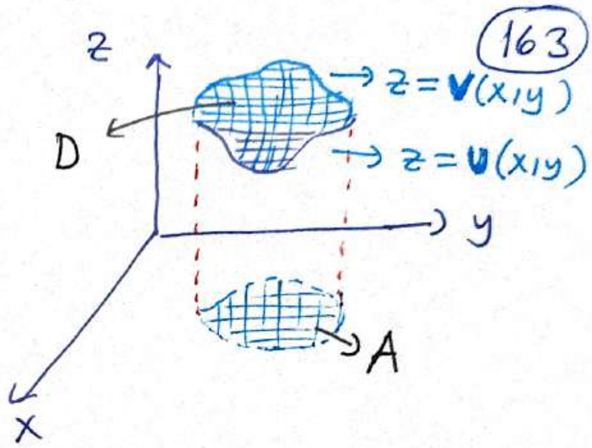
$$= \frac{5}{6}\,x^2\Big|_0^1$$

$$= \frac{5}{6}.$$

Ex. Find $\iiint_B x^2y^2z^2\,dV$, where

$B = \{(x,y,z) \in \mathbb{R}^3 \mid x \in [0,1],\ y \in [0,2],\ z \in [0,3]\}$

Triple Integral Calculation for a Solid Region Simple in z-Direction



Let D be a solid region bounded below by $z = u(x,y)$ and above by $z = v(x,y)$, projecting onto a region A in the xy-plane. If A is either type I or II, then the integral of a continuous function $f(x,y,z)$ over D is:

$$\boxed{\iiint_D f(x,y,z)\,dV = \iint_A \int_{u(x,y)}^{v(x,y)} f(x,y,z)\,dz\,dA.}$$

If A is type I, then $x \in [a,b]$, $y \in [g_1(x), g_2(x)]$,

if A is type II, then $x \in [h_1(y), h_2(y)]$, $y \in [c,d]$.

That is, $f = f(x,y,z)$,

$\iiint_D f(x,y,z)\,dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u(x,y)}^{v(x,y)} f.\,dz\,dy\,dx$	(type I)
$\iiint_D f(x,y,z)\,dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u(x,y)}^{v(x,y)} f.\,dz\,dx\,dy$	(type II)

If $f(x,y,z) = f_1(x).f_2(y).f_3(z)$, then

$$\iiint_B f(x,y,z)\,dV = \int_a^b f_1(x)\,dx \int_c^d f_2(y)\,dy \int_m^n f_3(z)\,dz,$$

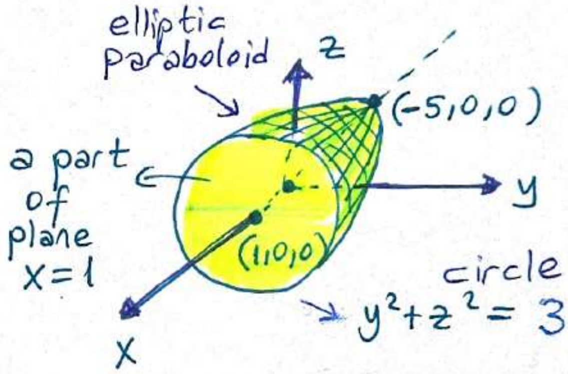
where $B = \{(x,y,z) \in \mathbb{R}^3 \mid x \in [a,b], y \in [c,d], z \in [m,n]\}$.

Ex. Evaluate $\iiint_E yz\,dV$ where E is the region bounded by $x = 2y^2 + 2z^2 - 5$ and the plane $x = 1$.

integrate first

$2y^2 + 2z^2 \le x \le 1$

D will be in the yz-plane



$$\iiint_E yz\,dV = \iint_D \left[\int_{2y^2+2z^2-5}^{1} yz\,dx\right] dA$$

$2y^2+2z^2-5=1$

$2y^2+2z^2=6$

$y^2+z^2=3$

intersection of the two surfaces

Then, $\boxed{y^2+z^2=3}$

$\Rightarrow \quad y^2+z^2=r^2 \qquad \begin{cases} y = r\sin\theta \\ z = r\cos\theta \end{cases} \qquad \begin{cases} \theta\in[0,2\pi] \\ r\in[0,\sqrt{3}] \end{cases}$

$$\Rightarrow \iiint_E yz\,dV = \iint_D (xyz)\Big|_1^{2y^2+2z^2-5} dA$$

$$= \iint_D \left[1-(2y^2+2z^2-5)\right] yz\,dA$$

$$= \iint_D \left[6-2(y^2+z^2)\right] yz\,dA$$

! It is not volume V $\uparrow$

$$= \int_0^{2\pi}\int_0^{\sqrt{3}} (6-2r^2)(r\sin\theta)(r\cos\theta)\,r\,dr\,d\theta$$

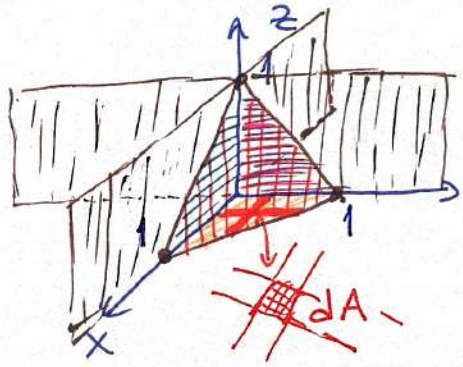
$$= \int_0^{2\pi} \left(\frac{3}{2}r^4 - \frac{1}{3}r^6\right)\sin\theta\cos\theta\Big|_0^{\sqrt{3}} d\theta$$

$$\iiint_E yz\,dV = \int_0^{2\pi} \frac{9}{2}\sin\theta\cos\theta\,d\theta = \int_0^{2\pi}\frac{9}{4}\sin 2\theta\,d\theta = 0.$$

Volume by Triple Integrals

$$\boxed{V = \iiint_D dV}$$ { volume of a solid in 3D }

EX. Find the volume V of the surface bounded by the plane $x+y+z=1$ and the planes $x=0, y=0, z=0$.



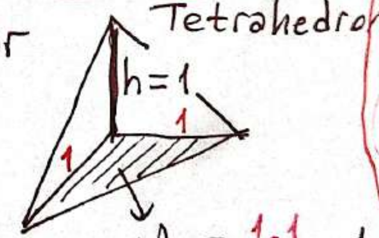
$z = 1-x-y$

$$V = \iiint_D dV$$

$$= \iint_A \int_0^{1-x-y} dz\, dA$$

$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz\, dy\, dx = \int_0^1 \int_0^{1-x} z \Big|_0^{1-x-y} dy\, dx$$

or



$A = \frac{1 \cdot 1}{2} = \frac{1}{2}$

$V = A \cdot h \cdot \frac{1}{3}$

$= \frac{1}{2} \cdot 1 \cdot \frac{1}{3}$

$V = \frac{1}{6}$

$$= \int_0^1 \int_0^{1-x} \left[(1-x-y) - 0\right] dy\, dx$$

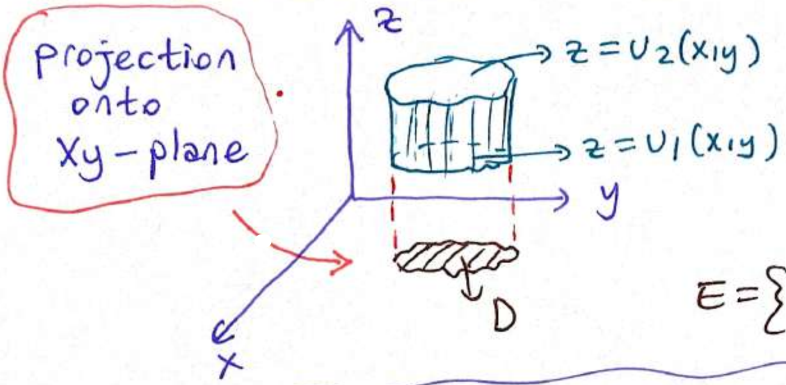
$$= \int_0^1 \left(y - xy - \frac{y^2}{2}\right)\Big|_0^{1-x} dx$$

$$= \int_0^1 \left[(1-x) - x(1-x) - \frac{(1-x)^2}{2} - 0\right] dx$$

$$= \int_0^1 \left[1 - x - x + x^2 - \frac{1}{2}(1 - 2x + x^2)\right] dx$$

$$= \int_0^1 \left(\frac{1}{2} - x + \frac{x^2}{2}\right) dx = \left(\frac{x}{2} - \frac{x^2}{2} + \frac{x^3}{6}\right)\Big|_0^1$$

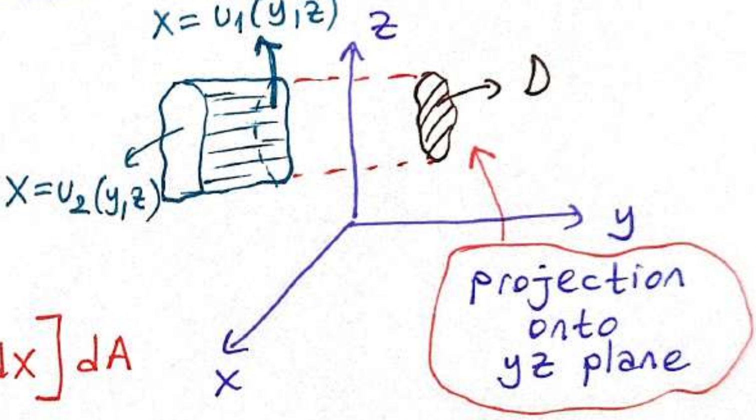
$$V = \cancel{\frac{1}{2}} - \cancel{\frac{1}{2}} + \frac{1}{6} \Rightarrow V = \frac{1}{6}$$



$$V = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} \underbrace{f(x,y,z)}_{=1} \, dz \right] dA$$

$$V = \iiint_E dV$$

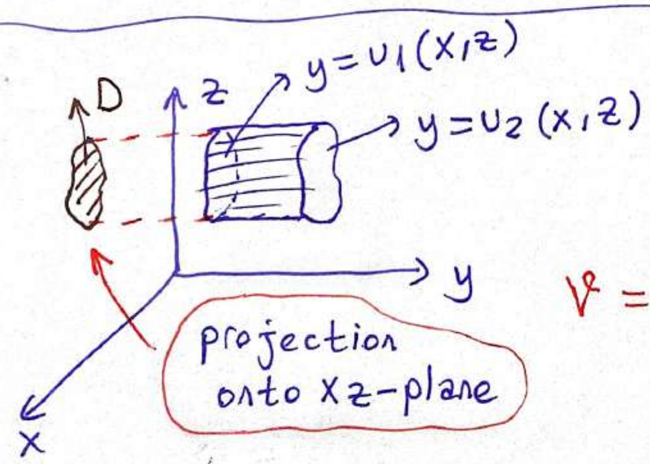
$$E = \{ (x,y,z) \mid (x,y) \in D,\ z \in [u_1(x,y), u_2(x,y)] \}$$



$$V = \iiint_E dv$$

$$V = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} \underbrace{f(x,y,z)}_{=1} \, dx \right] dA$$

$$E = \{ (x,y,z) \mid (y,z) \in D,\ x \in [u_1(y,z), u_2(y,z)] \}$$



$$V = \iiint_E f(x,y,z) \, dA$$

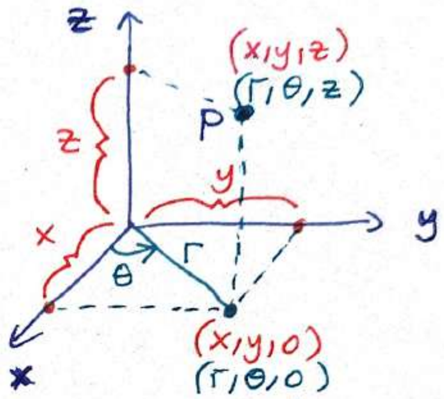
$$V = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} \underbrace{f(x,y,z)}_{=1} \, dy \right] dA$$

$$E = \{ (x,y,z) \mid (x,z) \in D,\ y \in [u_1(x,z), u_2(x,z)] \}$$

12.7 Cylindrical and Spherical Coordinates

Cylindirical Coordinates

Cylindrical coordinates extend polar coordinates to 3D by adding a vertical z-coordinate. A point $P(x,y,z)$ in rectangular coordinates is represented in cylindrical coordinates by its polar coordinates in the xy-plane along with the same z-value.



The cylindirical coordinates

Conversion Formulas:

from cylindrical to rectangular	from rectangular to cylindirical
$x = r\cos\theta$	$r = \sqrt{x^2+y^2}$
$y = r\sin\theta$	$\theta = \arctan\left(\frac{y}{x}\right)$
$z = z$	$z = z$

We can write the following surfaces from rectangular to cylindrical coordinates :

surface	rectangular equation	cylindrical equation
cylinder	$x^2+y^2=a^2$	$r=a$
cone	$x^2+y^2=z^2$	$r=z$
paraboloid	$x^2+y^2=az$	$r^2=az$
hyperboloid	$x^2+y^2-z^2=1$	$r^2=z^2+1$

The triple integral of a continuous function $f(x,y,z)$ over a solid region D in cylindrical coordinates is given by

$$\boxed{\iiint_D f(x,y,z)\,dV = \int_{\alpha}^{\beta}\int_{h_1(\theta)}^{h_2(\theta)}\int_{u(r,\theta)}^{v(r,\theta)} f(r\cos\theta, r\sin\theta, z)\, r\,dz\,dr\,d\theta.}$$

Ex. Find the equation in cylindrical coordinates for the elliptic paraboloid given by $z = 4x^2+y^2$.

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}\right\}$$ the relationships in cylindrical coordinates

$$z = 4x^2+y^2 \Rightarrow z = 4(r\cos\theta)^2 + (r\sin\theta)^2$$
$$z = 4r^2\cos^2\theta + r^2\sin^2\theta$$
$$z = 4r^2(1-\sin^2\theta) + r^2\sin^2\theta$$
$$z = r^2(4-3\sin^2\theta)$$

or

$$z = 4x^2+y^2 \Rightarrow z = 4(r\cos\theta)^2 + (r\sin\theta)^2$$
$$z = 4r^2\cos^2\theta + r^2\sin^2\theta$$
$$z = 4r^2\cos^2\theta + r^2(1-\cos^2\theta)$$
$$z = r^2(1+3\cos^2\theta)$$

Ex. Find the equation in cylindrical coordinates for the surfaces given by

a) $z = 3x^2+4y^2$

b) $z = x^2-y^2$

c) $2z = -2x^2+y^2$

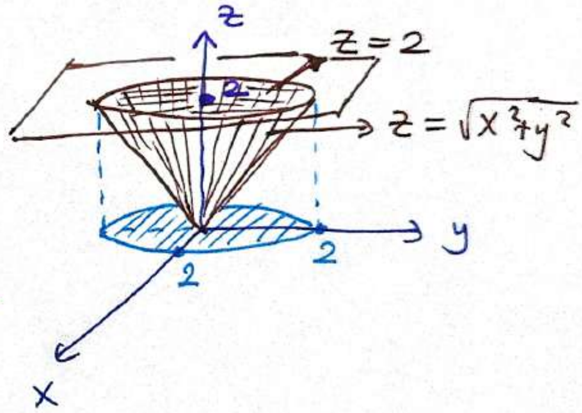
d) $\frac{z}{2} = 3x^2-4y^2$

Ex. Evaluate the triple integral

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)\,dz\,dy\,dx.$$

$E=\{(x,y,z) \mid x\in[-2,2], y\in[-\sqrt{4-x^2},\sqrt{4-x^2}], z\in[\sqrt{x^2+y^2},2]\}$

The projection of E onto xy-plane is the circle $x^2+y^2\leq 4$.



The lower surface of E is the cone $z=\sqrt{x^2+y^2}$ and the upper surface is the plane $z=2$.

The expression for the region E in cylindrical coordinates is much simplier:

$E=\{(r,\theta,z) \mid \theta\in[0,2\pi], r\in[0,2], z\in[r,2]\}$

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)\,dz\,dy\,dx$$

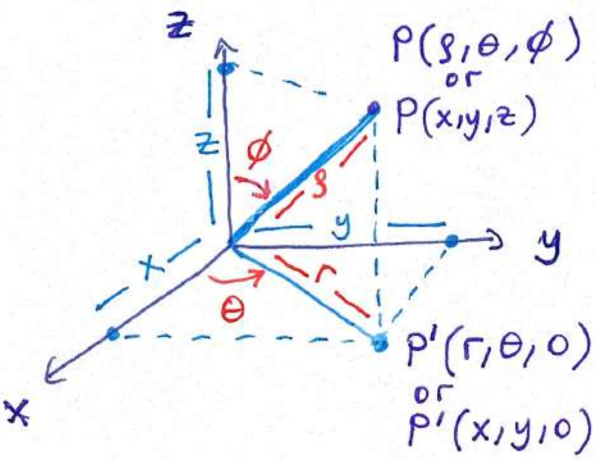
$$=\iiint_E (x^2+y^2)\,dV=\int_0^{2\pi}\int_0^{2}\int_r^{2} r^2\cdot r\,dz\,dr\,d\theta$$

$$=\int_0^{2\pi}d\theta\int_0^2 r^3(2-r)\,dr$$

$$=2\pi\left[\frac{1}{2}r^4-\frac{1}{5}r^5\right]\Big|_0^2$$

$$=\frac{16\pi}{5}.$$

Spherical Coordinates



Formulas:

Spherical to Rectangular

(ρ,θ,ϕ) to (x,y,z)

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho\cos\phi$$

(ρ: rho)

Spherical to Cylindrical

(ρ,θ,ϕ) to (r,θ,z)

$$r = \rho\sin\phi$$
$$\theta = \theta$$
$$z = \rho\cos\phi$$

Rectangular to Spherical

(x,y,z) to (ρ,θ,ϕ)

$$\rho = \sqrt{x^2+y^2+z^2}$$
$$\tan\theta = \frac{y}{x}$$
$$\phi = \arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$$

Cylindrical to Spherical

(r,θ,z) to (ρ,θ,ϕ)

$$\rho = r^2+z^2$$
$$\theta = \theta$$
$$\phi = \arccos\left(\frac{z}{\sqrt{r^2+z^2}}\right)$$

Ex. Write the following equations in spherical form.

a) $x^2+y^2+z^2 = c^2$, $c \geq 0$

b) $z = \sqrt{x^2+y^2}$

a) $\rho = \sqrt{x^2+y^2+z^2} \Rightarrow x^2+y^2+z^2 = \rho^2$

$\Rightarrow \rho^2 = c^2$

$\Rightarrow \rho = c \quad (c \geq 0)$

b) $\rho\cos\phi = \left[(\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2\right]^{1/2}$

$\Rightarrow \rho\cos\phi = \sqrt{\rho^2\sin^2\phi}$

$\Rightarrow \tan\phi = 1$ (or $\phi = \frac{\pi}{4}$)

Triple Integral in Spherical Coordinates

If $f(x,y,z)$ is continuous throughout the bounded solid region D, then

$$\iiint_D f(x,y,z)\,dV$$

$$= \iiint_{\bar{D}} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\underbrace{\rho^2\sin\phi}_{\text{Jacobian determinant}}\,d\rho\,d\theta\,d\phi .$$

Here, $\bar{D}$ represents the region D expressed in spherical coordinates, and Jacobian determinant describes

$$\left|\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}\right| = \rho^2\sin\phi .$$

Ex. Find the volume of the sphere using triple integral in spherical coordinates.

$\rho = a$ (equation of the sphere with radius $a \geq 0$)

for $\theta \in [0, 2\pi]$, $\phi \in [0, \pi]$.

$$\Rightarrow V = \iiint_D dV$$

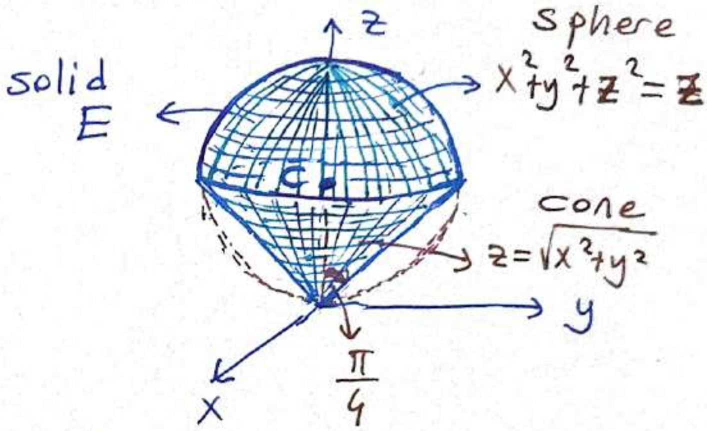
$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{\rho^3}{3}\sin\phi\,\Big|_{\rho=0}^{a}\,d\phi\,d\theta = \frac{a^3}{3}\int_0^{2\pi}\int_0^{\pi}\sin\phi\,d\phi\,d\theta$$

$$= \frac{a^3}{3}\int_0^{2\pi}(-\cos\phi)\Big|_0^{\pi}d\theta = \frac{2a^3}{3}\int_0^{2\pi}d\theta = \frac{2a^3}{3}\,\theta\Big|_0^{2\pi}$$

$\boxed{V = \frac{4}{3}\pi a^3}$ (volume of the sphere with radius a)

Ex. Find the volume of the solid above the cone $z=\sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2=z$ using spherical coordinates.



$$x^2+y^2+z^2-z=0$$

$$x^2+y^2+\left(z-\frac{1}{2}\right)^2=\frac{1}{4}$$

$$C\left(0,0,\frac{1}{2}\right),\ r=\frac{1}{2}$$

$$x^2+y^2+z^2=z \Rightarrow \rho^2=\rho\cos\phi$$

$$\Rightarrow \rho=\cos\phi \quad \left\{\begin{array}{l}\text{equation of the sphere}\\ \text{in spherical coordinates}\end{array}\right\}$$

$$\rho\cos\phi=\sqrt{\rho^2\sin^2\phi\cos^2\theta+\rho^2\sin^2\phi\sin^2\theta}=\rho\sin\phi$$

$$\Rightarrow \rho\cos\phi=\rho\sin\phi \Rightarrow \phi=\frac{\pi}{4}. \quad \left\{\begin{array}{l}\text{equation of the cone}\\ \text{in spherical coordinates}\end{array}\right\}$$

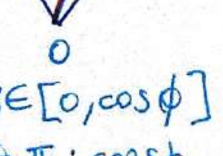
$$E=\left\{(\rho,\theta,\phi)\mid \theta\in[0,2\pi],\ \phi\in\left[0,\frac{\pi}{4}\right],\ \rho\in[0,\cos\phi]\right\}$$

$$\Rightarrow V=\iiint_E dV=\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\cos\phi}\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

$$=\int_0^{2\pi}d\theta\int_0^{\pi/4}\sin\phi\left(\frac{\rho^3}{3}\right)\Big|_0^{\cos\phi}d\phi$$

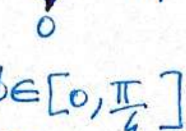
$$=\frac{2\pi}{3}\int_0^{\pi/4}\sin\phi\cos^3\phi\, d\phi=\frac{2\pi}{3}\left(-\frac{\cos^4\phi}{4}\right)\Big|_0^{\pi/4}$$

$$\Rightarrow V=\frac{\pi}{8}$$



$\rho\in[0,\cos\phi]$

θ,π : const.



$\phi\in[0,\frac{\pi}{4}]$

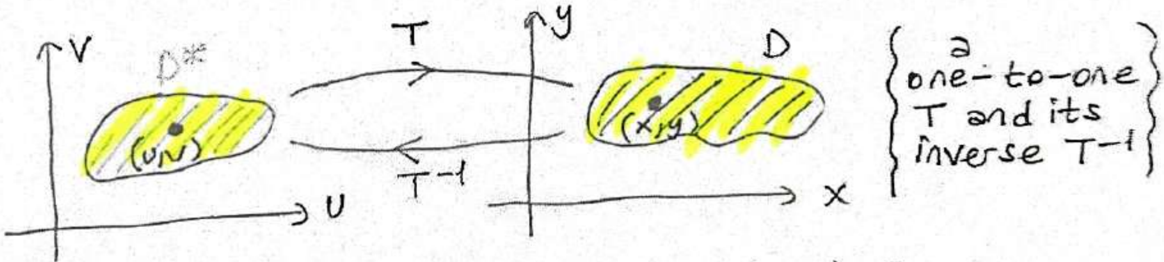
θ : const.

$\theta\in[0,2\pi]$

12.8. Jacobians : Change of Variables

With the substitution $x = g(u)$, the integral $\int_a^b f(x)\,dx$ becomes $\int_c^d f(g(u))\cdot g'(u)\,du$, where $a = g(c)$ and $b = g(d)$. The factor $g'(u)$ adjusts for the change of variable.

In a double integral $\iint_D f(x,y)\,dA$, changing variables transforms the integrand and region D to simplify evaluation. This requires a Jacobian factor, which adjusts for the area change under the new variables, similar to derivative in single-variable integrals.



The relationship between (x,y) and (u,v) is given by the transformation T:

$$x = g(u,v),\ y = h(u,v).\ \left(\text{or } x = x(u,v),\ y = y(u,v)\right)$$

from the uv-plane to xy-plane.

Here, $T(u,v) = (x,y)$

$$T: D^* \longrightarrow D$$
$$(u,v) \longrightarrow T(u,v) = (x,y) \quad \left\{\begin{array}{l}\text{transformation}\\ \text{from } D^* \text{ to } D\end{array}\right\}$$

$$T^{-1}: D \longrightarrow D^*$$
$$(x,y) \longrightarrow T^{-1}(x,y) = (u,v) \quad \left\{\begin{array}{l}\text{inverse}\\ \text{transformation}\\ \text{from } D \text{ to } D^*\end{array}\right\}$$

Let f be continuous on region D in the xy-plane, and T a one-to-one transformation mapping D^* in the uv-plane onto D. The change of variables $x=g(u,v)$, $y=h(u,v)$ introduces a Jacobian $J(u,v)$, which is nonzero and doesn't change sign on D^*, representing the area change in the transformation. Then,

absolute value

$$\boxed{\iint_D f(x,y)\,dy\,dx = \iint_{D^*} f[g(u,v),h(u,v)]\,\big|\,|J(u,v)|\,\big|\,du\,dv ,}$$

{determinant}

where

$$|J(u,v)| = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

$$\Rightarrow \boxed{|J(u,v)| = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{\partial x}{\partial u}\cdot\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\cdot\frac{\partial y}{\partial u}} \quad \left\{ \begin{array}{c} \text{or} \\ x_u\cdot y_v - x_v\cdot y_u \end{array} \right\}$$

Ex. Determine the Jacobian for the transformation from rectangular to polar coordinates, where

$$x = r\cos\theta , \quad y = r\sin\theta .$$

$$|J(r,\theta)| = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= (\cos\theta)\cdot(r\cos\theta) - (\sin\theta)\cdot(-r\sin\theta)$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r(\cos^2\theta + \sin^2\theta)$$

$$\boxed{|J(r,\theta)| = \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = r.} \quad \left\{ \iint_D f(x,y)\,dy\,dx = \iint_{D^*} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta \right\}$$

The general Jacobian formula states that if a general transformation is made with u and v expressed in terms of x and y, the Jacobian determinant can be computed by

$$\boxed{\left|\frac{\partial(u,v)}{\partial(x,y)}\right| = \frac{1}{\left|\frac{\partial(x,y)}{\partial(u,v)}\right|}} \qquad \left\{ \left|\frac{\partial(u,v)}{\partial(x,y)}\right| \cdot \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = 1 \right\}$$

Ex. Let $u = x^2 + y^2$ and $v = xy$. Compute the Jacobian $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$.

$$\left|\frac{\partial(u,v)}{\partial(x,y)}\right| = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

$$= \begin{vmatrix} 2x & 2y \\ y & x \end{vmatrix} = (2x)(x) - (y)(2y)$$

$$\Rightarrow \left|\frac{\partial(u,v)}{\partial(x,y)}\right| = 2(x^2 - y^2). \text{ Then,}$$

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{\left|\frac{\partial(u,v)}{\partial(x,y)}\right|} = \frac{1}{2(x^2 - y^2)} = \underline{\frac{1}{2\sqrt{u^2 - 4v^2}}}$$

$$(x^2+y^2)^2 - 4x^2y^2 = (x^2 - y^2)^2$$

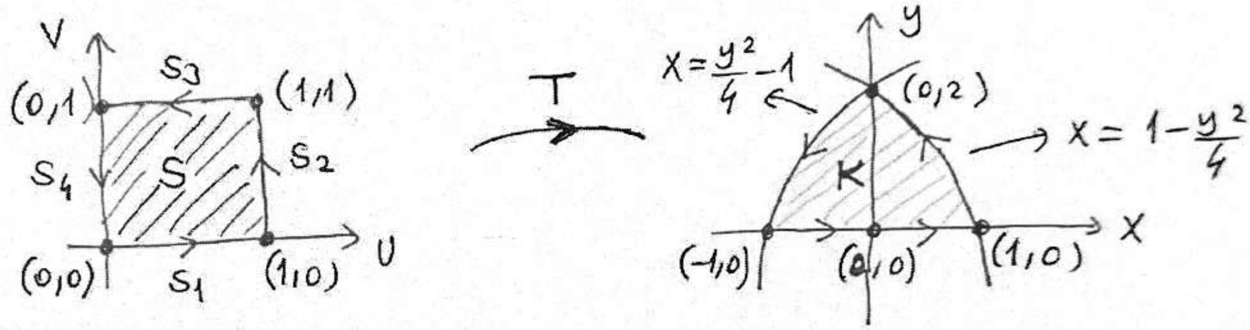
$$\Rightarrow u^2 - 4v^2 = (x^2 - y^2)^2 \Rightarrow x^2 - y^2 = \sqrt{u^2 - 4v^2}.$$

Ex. A transformation is defined by

$$x = u^2 - v^2 \quad \text{and} \quad y = 2uv.$$

Find the image of the square

$$S = \{(u,v) \mid u \in [0,1], v \in [0,1]\}.$$



The transformation sends the boundary of S to the boundary of the image. Then, we start by finding the image of the edges of S.

The side of S_1 is given as $v=0$ $(0 \leq u \leq 1)$. From the given equations $x = u^2$, $y=0$ and hence $0 \leq x \leq 1$. Thus, the image of S_1 is the line segment from $(0,0)$ to $(1,0)$.

The side of S_2 is $u=1$ $(0 \leq v \leq 1)$, and by putting $u=1$ in given equations

$$x = 1 - v^2 \quad \text{and} \quad y = 2v.$$

By eliminating v, we find that a parabola is a segment of

$$x = 1 - \frac{y^2}{4} \quad (0 \leq x \leq 1).$$

Similarly for S_3: $v=1$ $(0 \leq u \leq 1)$

$$x = \frac{y^2}{4} - 1 \quad (-1 \leq x \leq 0).$$

For S_4: $u=0$ $(0 \le v \le 1)$,

$$x = -v^2, \quad y = 0.$$

Note that when we move counterclockwise around the square, we also move counterclockwise around the parabolic region.

Ex. Calculate $\iint_R e^{\frac{x+y}{x-y}}\, dA$, where R is a trapezoidal region with vertices $(1,0), (2,0), (0,-2)$ and $(0,-1)$.

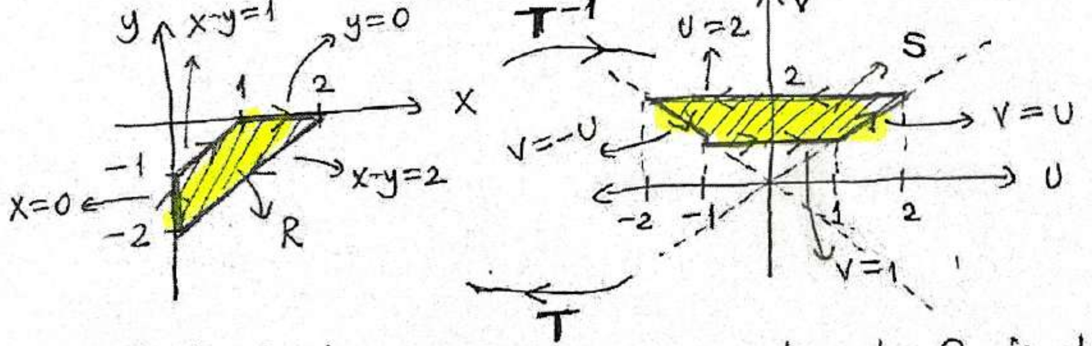
(1)... $u = x+y$ and $v = x-y$ $\quad \left(T(u,v) = (x,y)\right)$

$\Rightarrow$ $x = \frac{1}{2}(u+v)$, $y = \frac{1}{2}(u-v)$. $\quad \left(T^{-1}(x,y) = (u,v)\right)$

(2)...

Jacobian of T:

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$



To find the region S corresponding to R in the uv-plane the edges of R:

$$y=0, \quad x-y=2, \quad x=0, \quad x-y=1.$$

Note that the image lines in the uv-plane are on the lines, from (1) or (2), we find

$$u = v \;,\; v = 2 \;,\; u = -v \;,\; v = 1 \,.$$

Therefore the region S is the trapezoidal region with vertices $(1,1), (2,2), (-2,2)$, and $(-1,1)$.

$$S = \{ (u,v) \mid v \in [1,2] ,\, u \in [-v, v] \} .$$

$$\Rightarrow \iint_R e^{\frac{x+y}{x-y}} \, dA = \iint_S e^{\frac{u}{v}} \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

$$= \int_1^2 \int_{-v}^{v} e^{\frac{u}{v}} \left(\tfrac{1}{2}\right) du\,dv \quad \left\{ \left|-\tfrac{1}{2}\right| \text{ absolute value} \right\}$$

$$= \frac{1}{2} \int_1^2 v\, e^{\frac{u}{v}} \Big|_{u=-v}^{u=v} dv$$

$$= \frac{1}{2} \int_1^2 (e - e^{-1})\, v \, dv$$

$$\Rightarrow \iint_R e^{\frac{x+y}{x-y}} \, dA = \frac{3}{4}(e - e^{-1})$$

There is a similar formula for change of variables for triple integrals.

The transformation T is a region S in uvw-space that projects onto a region R in xyz-space, via

$$x = g(u,v,w) \;,\; y = h(u,v,w) \;,\; z = k(u,v,w) \,,$$

Jacobian of T is given by

$$\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}_{3\times 3}.$$

Then, for the triple integral

$$\iiint_R f(x,y,z)\,dV$$

$$= \iiint_S f\big(x(u,v,w), y(u,v,w), z(u,v,w)\big) \left|\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|\right| du\,dv\,dw$$

{absolute value of the determinant}

Ex. Derive the formula for triple integration in spherical coordinates.

$x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$, $z = \rho\cos\phi$

$$\Rightarrow \left|\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}\right| = \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix}$$

$$= \cos\phi \cdot \begin{vmatrix} -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \end{vmatrix}$$

$$- \rho\sin\phi \cdot \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix}$$

$$= \cos\phi\left(-\rho^2\sin\phi\cos\phi\sin^2\theta - \rho^2\sin\phi\cos\phi\cos^2\theta\right)$$
$$- \rho\sin\phi\left(\rho\sin^2\phi\cos^2\theta + \rho\sin^2\phi\sin^2\theta\right)$$
$$= -\rho^2\sin\phi\cos^2\phi - \rho^2\sin\phi\sin^2\phi = -\rho^2\sin\phi$$

$$\boxed{\left|\left|\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}\right|\right| = \rho^2\sin\phi}$$ (we use absolute value)

$$\Rightarrow \iiint_R f(x,y,z)\,dV$$

$$= \iiint_S f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

Ex. Compute $\iint_R \frac{1}{(x-y)^2}\,dA$ on region defined by

$R = \{(x,y) \mid x \in [0,1],\ y \in [2,4]\}$.

Ex. Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} (x^2+y^2+z^2)^2\,dz\,dy\,dx$

in spherical coordinates.

Ex. Find the volume

$$V = \int_0^{2\pi} \int_0^{\pi/6} \int_1^3 \rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$$

Ex. Compute

$$\int_0^{\sqrt{2}} \int_y^{\sqrt{4-y^2}} \frac{1}{1+x^2+y^2}\,dx\,dy$$

by using the polar coordinates.

Ex. Evaluate

$$\int_0^{\pi} \int_1^{1+\sin\theta} r\,dr\,d\theta.$$